Quadratics - Quadratic in Form

Objective: Solve equations that are quadratic in form by substitution to create a quadratic equation.

We have seen three different ways to solve quadratics: factoring, completing the square, and the quadratic formula. A quadratic is any equation of the form $0 = ax^2 + bx + c$, however, we can use the skills learned to solve quadratics to solve problems with higher (or sometimes lower) powers if the equation is in what is called quadratic form.

Quadratic Form: $\mathbf{0 = ax^m + bx^n + c}$ **where** $\mathbf{m = 2n}$

An equation is in quadratic form if one of the exponents on a variable is double the exponent on the same variable somewhere else in the equation. If this is the case we can create a new variable, set it equal to the variable with smallest exponent. When we substitute this into the equation we will have a quadratic equation we can solve.

World View Note: Arab mathematicians around the year 1000 were the first to use this method!

Example 1.

$x^4 - 13x^2 + 36 = 0$	Quadratic form, one exponent, 4, double the other, 2
$y = x^2$	New variable equal to the variable with smaller exponent
$y^2 = x^4$	Square both sides
$y^2 - 13y + 36 = 0$	Substitute y for x^2 and y^2 for x^4
$(y - 9)(y - 4) = 0$	Solve. We can solve this equation by factoring
$y - 9 = 0$ or $y - 4 = 0$	Set each factor equal to zero
$+9 + 9$ $\quad$ $+4 + 4$	Solve each equation
$y = 9$ or $y = 4$	Solutions for y, need x. We will use $y = x^2$ equation
$9 = x^2$ or $4 = x^2$	Substitute values for y
$\pm\sqrt{9} = \sqrt{x^2}$ or $\pm\sqrt{4} = \sqrt{x^2}$	Solve using the even root property, simplify roots
$x = \pm 3, \pm 2$	Our Solutions

When we have higher powers of our variable, we could end up with many more solutions. The previous equation had four unique solutions.

Example 2.

$a^{-2} - a^{-1} - 6 = 0$	Quadratic form, one exponent, -2, is double the other, -1
$b = a^{-1}$	Make a new variable equal to the variable with lowest exponent
$b^2 = a^{-2}$	Square both sides
$b^2 - b - 6 = 0$	Substitute b^2 for a^{-2} and b for a^{-1}
$(b-3)(b+2) = 0$	Solve. We will solve by factoring
$b - 3 = 0$ or $b + 2 = 0$	Set each factor equal to zero
$\underline{+3+3 \qquad -2-2}$	Solve each equation
$b = 3$ or $b = -2$	Solutions for b, still need a, substitute into $b = a^{-1}$
$3 = a^{-1}$ or $-2 = a^{-1}$	Raise both sides to -1 power
$3^{-1} = a$ or $(-2)^{-1} = a$	Simplify negative exponents
$a = \frac{1}{3}, -\frac{1}{2}$	Our Solution

Just as with regular quadratics, these problems will not always have rational solutions. We also can have irrational or complex solutions to our equations.

Example 3.

$2x^4 + x^2 = 6$	Make equation equal to zero
$\underline{-6-6}$	Subtract 6 from both sides
$2x^4 + x^2 - 6 = 0$	Quadratic form, one exponent, 4, double the other, 2
$y = x^2$	New variable equal variable with smallest exponent
$y^2 = x^4$	Square both sides
$2y^2 + y - 6 = 0$	Solve. We will factor this equation
$(2y-3)(y+2) = 0$	Set each factor equal to zero
$2y - 3 = 0$ or $y + 2 = 0$	Solve each equation
$\underline{+3+3 \qquad -2-2}$	
$\frac{2y}{2} = \frac{3}{2}$ or $y = -2$	
$y = \frac{3}{2}$ or $y = -2$	We have y, still need x. Substitute into $y = x^2$
$\frac{3}{2} = x^2$ or $-2 = x^2$	Square root of each side
$\pm\sqrt{\frac{3}{2}} = \sqrt{x^2}$ or $\pm\sqrt{-2} = \sqrt{x^2}$	Simplify each root, rationalize denominator
$x = \frac{\pm\sqrt{6}}{2}, \pm i\sqrt{2}$	Our Solution

When we create a new variable for our substitution, it won't always be equal to just another variable. We can make our substitution variable equal to an expression as shown in the next example.

Example 4.

$$
\begin{array}{rl}
3(x-7)^2-2(x-7)+5=0 & \text{Quadratic form} \\
y=x-7 & \text{Define new variable} \\
y^2=(x-7)^2 & \text{Square both sides} \\
3y^2-2y+5=0 & \text{Substitute values into original} \\
(3y-5)(y+1)=0 & \text{Factor} \\
3y-5=0 \ \text{ or } \ y+1=0 & \text{Set each factor equal to zero} \\
\underline{+5+5} \qquad \underline{-1-1} & \text{Solve each equation} \\
3y=5 \ \text{ or } \ y=-1 & \\
\overline{3} \quad \overline{3} & \\
y=\frac{5}{3} \ \text{ or } \ y=-1 & \text{We have } y\text{, we still need } x. \\
\frac{5}{3}=x-7 \ \text{ or } \ -1=x-7 & \text{Substitute into } y=x-7 \\
\underline{+\frac{21}{3} \qquad +7} \qquad \underline{+7 \qquad +7} & \text{Add 7. Use common denominator as needed} \\
x=\frac{26}{3},6 & \text{Our Solution}
\end{array}
$$

Example 5.

$$
\begin{array}{rl}
(x^2-6x)^2=7(x^2-6x)-12 & \text{Make equation equal zero} \\
-7(x^2-6x)+12-7(x^2-6x)+12 & \text{Move all terms to left} \\
(x^2-6x)^2-7(x^2-6x)+12=0 & \text{Quadratic form} \\
y=x^2-6x & \text{Make new variable} \\
y^2=(x^2-6x)^2 & \text{Square both sides} \\
y^2-7y+12=0 & \text{Substitute into original equation} \\
(y-3)(y-4)=0 & \text{Solve by factoring} \\
y-3=0 \ \text{ or } \ y-4=0 & \text{Set each factor equal to zero} \\
\underline{+3+3} \qquad \underline{+4+4} & \text{Solve each equation} \\
y=3 \ \text{ or } \ y=4 & \text{We have } y\text{, still need } x. \\
3=x^2-6x \ \text{ or } \ 4=x^3-6x & \text{Solve each equation, complete the square} \\
\left(\frac{1}{2}\cdot 6\right)^2=3^2=9 & \text{Add 9 to both sides of each equation} \\
12=x^2-6x+9 \ \text{ or } \ 13=x^2-6x+9 & \text{Factor} \\
12=(x-3)^2 \ \text{ or } \ 13=(x-3)^2 & \text{Use even root property} \\
\pm\sqrt{12}=\sqrt{(x-3)^2} \ \text{ or } \ \pm\sqrt{13}=\sqrt{(x-3)^2} & \text{Simplify roots}
\end{array}
$$

$$\pm 2\sqrt{3} = x - 3 \ \text{ or } \ \pm\sqrt{13} = x - 3 \qquad \text{Add 3 to both sides}$$
$$\underline{+3 \quad +3} \qquad \underline{+3 \quad +3}$$
$$x = 3 \pm 2\sqrt{3}, 3 \pm \sqrt{13} \qquad \text{Our Solution}$$

The higher the exponent, the more solution we could have. This is illustrated in the following example, one with six solutions.

Example 6.

$x^6 - 9x^3 + 8 = 0$	Quadratic form, one exponent, 6, double the other, 3
$y = x^3$	New variable equal to variable with lowest exponent
$y^2 = x^6$	Square both sides
$y^2 - 9y + 8 = 0$	Substitute y^2 for x^6 and y for x^3
$(y - 1)(y - 8) = 0$	Solve. We will solve by factoring.
$y - 1 = 0$ or $y - 8 = 0$	Set each factor equal to zero
$\underline{+1+1} \qquad \underline{+8+8}$	Solve each equation
$y = 1$ or $y = 8$	Solutions for y, we need x. Substitute into $y = x^3$
$x^3 = 1$ or $x^3 = 8$	Set each equation equal to zero
$\underline{-1-1} \qquad \underline{-8-8}$	
$x^3 - 1 = 0$ or $x^3 - 8 = 0$	Factor each equation, difference of cubes
$(x - 1)(x^2 + x + 1) = 0$	First equation factored. Set each factor equal to zero
$x - 1 = 0$ or $x^2 + x + 1 = 0$	First equation is easy to solve
$\underline{+1+1}$	
$x = 1$	First solution
$\dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm i\sqrt{3}}{2}$	Quadratic formula on second factor
$(x - 2)(x^2 + 2x + 4) = 0$	Factor the second difference of cubes
$x - 2 = 0$ or $x^2 + 2x + 4 = 0$	Set each factor equal to zero.
$\underline{+2+2}$	First equation is easy to solve
$x = 2$	Our fourth solution
$\dfrac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = -1 \pm i\sqrt{3}$	Quadratic formula on second factor
$x = 1, 2, \dfrac{1 \pm i\sqrt{3}}{2}, -1 \pm i\sqrt{3}$	Our final six solutions

9.6 Practice - Quadratic in Form

Solve each of the following equations. Some equations will have complex roots.

1) $x^4 - 5x^2 + 4 = 0$

2) $y^4 - 9y^2 + 20 = 0$

3) $m^4 - 7m^2 - 8 = 0$

4) $y^4 - 29y^2 + 100 = 0$

5) $a^4 - 50a^2 + 49 = 0$

6) $b^4 - 10b^2 + 9 = 0$

7) $x^4 - 25x^2 + 144 = 0$

8) $y^4 - 40y^2 + 144 = 0$

9) $m^4 - 20m^2 + 64 = 0$

10) $x^6 - 35x^3 + 216 = 0$

11) $z^6 - 216 = 19z^3$

12) $y^4 - 2y^2 = 24$

13) $6z^4 - z^2 = 12$

14) $x^{-2} - x^{-1} - 12 = 0$

15) $x^{\frac{2}{3}} - 35 = 2x^{\frac{1}{3}}$

16) $5y^{-2} - 20 = 21y^{-1}$

17) $y^{-6} + 7y^{-3} = 8$

18) $x^4 - 7x^2 + 12 = 0$

19) $x^4 - 2x^2 - 3 = 0$

20) $x^4 + 7x^2 + 10 = 0$

21) $2x^4 - 5x^2 + 2 = 0$

22) $2x^4 - x^2 - 3 = 0$

23) $x^4 - 9x^2 + 8 = 0$

24) $x^6 - 10x^3 + 16 = 0$

25) $8x^6 - 9x^3 + 1 = 0$

26) $8x^6 + 7x^3 - 1 = 0$

27) $x^8 - 17x^4 + 16 = 0$

28) $(x-1)^2 - 4(x-1) = 5$

29) $(y+b)^2 - 4(y+b) = 21$

30) $(x+1)^2 + 6(x+1) + 9 = 0$

31) $(y+2)^2 - 6(y+2) = 16$

32) $(m-1)^2 - 5(m-1) = 14$

33) $(x-3)^2 - 2(x-3) = 35$

34) $(a+1)^2 + 2(a-1) = 15$

35) $(r-1)^2 - 8(r-1) = 20$

36) $2(x-1)^2 - (x-1) = 3$

37) $3(y+1)^2 - 14(y+1) = 5$

38) $(x^2-3)^2 - 2(x^2-3) = 3$

39) $(3x^2-2x)^2 + 5 = 6(3x^2-2x)$

40) $(x^2+x+3)^2 + 15 = 8(x^2+x+3)$

41) $2(3x+1)^{\frac{2}{3}} - 5(3x+1)^{\frac{1}{3}} = 88$

42) $(x^2+x)^2 - 8(x^2+x) + 12 = 0$

43) $(x^2+2x)^2 - 2(x^2+2x) = 3$

44) $(2x^2+3x)^2 = 8(2x^2+3x) + 9$

45) $(2x^2-x)^2 - 4(2x^2-x) + 3 = 0$

46) $(3x^2-4x)^2 = 3(3x^2-4x) + 4$

Answers - Quadratic in Form

1) $\pm 1, \pm 2$
2) $\pm 2, \pm \sqrt{5}$
3) $\pm i, \pm 2\sqrt{2}$
4) $\pm 5, \pm 2$
5) $\pm 1, \pm 7$
6) $\pm 3, \pm 1$
7) $\pm 3, \pm 4$
8) $\pm 6, \pm 2$
9) $\pm 2, \pm 4$
10) $2, 3, -1 \pm i\sqrt{3}, \frac{-3 \pm 3i\sqrt{3}}{2}$
11) $-2, 3, 1 \pm i\sqrt{3}, \frac{-3 \pm i\sqrt{3}}{2}$
12) $\pm \sqrt{6}, \pm 2i$
13) $\frac{\pm 2i\sqrt{3}}{3}, \frac{\pm \sqrt{6}}{2}$
14) $\frac{1}{4}, -\frac{1}{3}$
15) $-125, 343$
16) $-\frac{5}{4}, \frac{1}{5}$
17) $1, -\frac{1}{2}, \frac{1 \pm i\sqrt{3}}{4}, \frac{-1 \pm i\sqrt{3}}{2}$
18) $\pm 2, \pm \sqrt{3}$
19) $\pm i, \pm \sqrt{3}$
20) $\pm i\sqrt{5}, \pm i\sqrt{2}$
21) $\pm \sqrt{2}, \pm \frac{\sqrt{2}}{2}$
22) $\pm i, \frac{\pm 6}{2}$
23) $\pm 1, \pm 2\sqrt{2}$
24) $2, \sqrt[3]{2}, -1 \pm i\sqrt{3}, \frac{-\sqrt[3]{2} \pm i\sqrt[6]{108}}{2}$
25) $1, \frac{1}{2}, \frac{-1 \pm i\sqrt{3}}{4}, \frac{-1 \pm i\sqrt{3}}{2}$
26) $\frac{1}{2}, -1, \frac{-1 \pm i\sqrt{3}}{4}, \frac{1 \pm i\sqrt{3}}{2}$
27) $\pm 1, \pm i, \pm 2, \pm 2i$
28) $6, 0$
29) $-(b+3), 7-b$
30) -4
31) $-4, 6$
32) $8, -1$
33) $-2, 10$
34) $2, \ -6$
35) $-1, 11$
36) $\frac{5}{2}, 0$
37) $4, \ -\frac{4}{3}$
38) $\pm \sqrt{6}, \pm \sqrt{2}$
39) $\pm 1, -\frac{1}{3}, \frac{5}{3}$
40) $0, \pm 1, -2$
41) $\frac{511}{3}, -\frac{1339}{24}$
42) $-3, \pm 2, 1$
43) $\pm 1, -3$
44) $-3, -1, \frac{3}{2}, -\frac{1}{2}$
45) $\pm 1, -\frac{1}{2}, \frac{3}{2}$
46) $1, 2, \frac{1}{3}, -\frac{2}{3}$